Numbers, Infinity and Shapes

Introduction

Geometry is, perhaps, one of the most practical aspects of mathematics given its relevance to the world in which we live. Whilst it began as a need to measure shapes and define the relations between them, it now has important applications in a wide range of disciplines including navigation, construction, graphic design, fashion design, and robotics. Different mathematical numbers form the basis of many concepts in geometry, with none more fundamental than π (pi) and its myriad of properties. However, the true meaning of π is often hidden behind a button on a calculator and many do not realise that it is actually the ratio of a circle's circumference to its diameter.

A topic that is closely associated with geometry is topology; the study of relations between shapes and spaces. In fact, the **Möbius** strip is perhaps one of the most famous topological shapes and is unusual in that it is one-dimensional yet exists in a three-dimensional world.

Aim of workshop

The aim of this workshop is to instil a newfound appreciation for the number π and its wide range of applications and properties. In particular, this workshop hopes to encourage experiential learning of mathematics as students actively work together to approximate the value of π using only common everyday materials. The 'Buffon's Needles' trial will similarly be carried out to show students that π can be found in the most unexpected of places. Additionally, this workshop aims to introduce the topic of topology through the construction of a Möbius strip and exploration of its properties.

Learning Outcomes

By the end of this workshop students will be able to:

- Name various mathematical constants

- Derive the relationship between the diameter and circumference of a circle

- Recognise and apply π as a ratio

- Identify and construct a shape with only one face and one edge

Materials and Resources

Party hats (or any other circular based object), string, rulers/metre sticks, scissors, paper, Sellotape, 100 assorted wooden sticks (the size of matchsticks) per group

Key Words

Topology
Topology is the mathematical study of geometrical properties and spatial relations that are preserved through the deformation of objects

Möbius strip
A Möbius strip is a surface with only one face and one edge, formed by rotating one end of a rectangular strip of paper 180° about the longitudinal axis and attaching this end to the other to form a continuous loop

Numbers, Infinity and Shapes: Workshop Outline

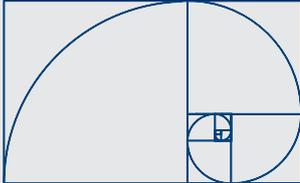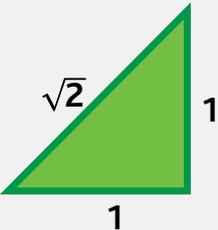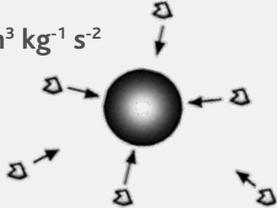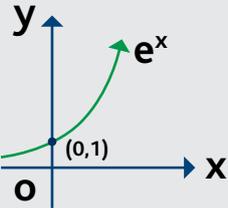| SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:05)	**Introduction to Famous Numbers**	– Ask students to write down as many important/famous numbers as they can. – Discuss answers as a class (see **Appendix – Note 1**). – Highlight the importance of π in geometry.
5 mins (00:10)	**Geometry Discussion**	– You may wish to ask students what they know about the history of geometry or its applications. – Provide students with a brief background on the topic (see **Appendix – Note 2**).
15 mins (00:25)	**Activity 1 Calculating π**	– Divide students into groups of 3 and ask them to think of a team name using either a famous number or mathematician e.g. Apple πs, Terrific Trapezoid, the Quaternions etc. – Provide each group with a party hat (or substitute object), ruler & string. – **Activity Sheet 1**: In their groups, students attempt to calculate π using only the above materials (i.e. students can use the string to find a value for the circumference of the circle, radius of the circle etc.) – Once everyone has finished, write down each group's value for π on the board. (Note: A pre-prepared spreadsheet projected onto the board may help with this). Which team got the most accurate value for π? What is the average for the class? – Ask students what other circular objects they used and what values they got for π.

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
15 mins (00:40)	**Activity 2 Buffon's Needles**	– Hand out 100 wooden sticks and the Buffon's Needles template sheet to each group and explain the task. – **Activity Sheet 2**: In their groups, students try to approximate π using the wooden sticks. Students should throw the wooden sticks randomly on the piece of paper and then fill in the summary table on the activity sheet (Questions you might ask during the activity and further details are available in **Appendix – Note 3**). – Write down each group's value for π on the board. What is the average for the class? Which team got the most accurate value for π? – Mention the work of the Chudnovsky Brothers (see **Appendix – Note 4**).
10 mins (00:50)	**Activity 3 Shapes**	– Ask the class "What is a shape?", "How do we categorise shapes?" (By the number of faces and edges they have). – Hand out scissors, paper and Sellotape. – **Activity Sheet 3:** Students are asked to draw and construct shapes with a specified number of faces and edges (see **Appendix – Note 5**).
5 mins (00:55)	**Constructing the Möbius Strip**	– Demonstrate how to construct a Möbius strip (see video link in **Additional Resources**) and ask students to make their own (students can now attempt the rest of Activity 3 if they haven't already done so).

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
10 mins (01:05)	**Activity 4** **Möbius Strip**	– **Activity 4**: Students try to predict what will happen to their Möbius strip when it is cut (see **Appendix – Note 6**).
15 mins (01:20)	**Activity 5** **Möbius Strip (Optional)**	– Hand out extra pieces of paper. – **Activity 5**: Students make strips with varying numbers of half loops and attempt to identify a pattern (see **Appendix – Note 7**).
5 mins (01:25)	**Klein Bottle**	– Ask students "What do you think would happen if you stick two Möbius strips along their edges?" – Describe the Klein Bottle (see **Appendix – Note 8**). – You may wish to show students the Klein Bottle video clip (see link in **Additional Resources**).
5 mins (01:30)	**The Doughnut and the Coffee Cup**	– Describe what is meant by topology and ask students how a coffee cup may be considered the same (topologically) as a doughnut before explaining it to the class (see **Appendix – Note 9**). – Ask students if a doughnut and a pretzel would be the same from a topological point of view.

Numbers, Infinity and Shapes – Workshop Appendix

Note 1: Important and Famous Numbers

Below are some of the most important and famous numbers in mathematics

Mathematical Constant	**The Golden Ratio**
1 number line from -5 to 5 with point at 1	$\varphi \approx 1.6180...$ golden ratio spiral diagram
Pythagoras' constant	**Newton's Gravitational Constant**
$\sqrt{2} \approx 1.41421356...$ $1^2 + 1^2 = \sqrt{2}^2$ right triangle with sides $\sqrt{2}$, 1, 1	$G \approx 6.67408 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Euler's number e	**Archimedes' constant**
$e \approx 2.71828...$ $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ graph of e^x through (0,1)	$\pi \approx 3.14159...$ circle with circumference c, diameter d, $\pi = \frac{c}{d}$

Note 2: Geometry Discussion

For the class discussion on Geometry, you may wish to mention some of the following points:

- The word geometry stems from the Greek word 'Geo' meaning earth and 'metria' meaning measure - therefore, it is the 'measure of the earth'

- **Eratosthenes** (276 - 194 BC) was chief librarian at the Great Library of Alexandria in Egypt and made an amazing approximation of the circumference of the Earth using a few measurements and simple, yet brilliant, geometry

- **Euclid** had, perhaps, the most profound impact on the development of geometry and his book 'The Elements' has become one of the most important and influential texts in the history of mathematics

- There are many different types of geometry including Euclidean, non-Euclidean, differential etc.

Geometry is a very practical area that has many important applications in today's world including molecular modelling, architecture, engineering, biology, fashion design and graphic design (this list is by no means exhaustive!)

Note 3: Solutions for Activity 2

The Buffon's Needle experiment is an interesting and unusual method for approximating the value of π without the need for circles or spheres. It involves dropping needles on a sheet of lined paper and calculating the proportion of needles that intersect with a line. This experiment was first posed by **Georges-Louis Leclerc,** Comte de Buffon, in the 18th century and is therefore named 'Buffon's Needles' in his honour.

Figure 1: Post-primary students working on the Buffon's Needles activity during a UCD Maths Sparks workshop in 2017

Q1. In your groups, throw the wooden sticks randomly on the piece of paper and then fill in the summary table below.

Students may wish to throw the wooden sticks in groups of 20 or 50 rather than all at once. Note: For this activity, it is important that the length of the wooden sticks be approximately half the distance between the lines on the template (see Figure 1 for example).

Q2. Using this formula, what value did you get for π?

Expect values between 1 and 5

Q3. How accurate is your value for π?

Values between 2.9 and 3.3 would be very good!

Q4. How could you make your value for π more accurate? Any suggestions?

- Take several measurements and calculate the average
- Use a range of different sized circles and calculate the average

Q5. Could we ever measure an exact value for π? Explain your reasoning.

No, it is impossible! π is an irrational number and cannot be expressed as a fraction – it has no ending digit, nor does it contain a repeating decimal pattern!

Note 4: David and Gregory Chudnovsky

Mathematicians and brothers, **David and Gregory Chudnovsky,** are renowned for building a supercomputer (known as m-zero) in their tiny apartment in Manhattan in an effort to explore the properties of π. Whilst no evident pattern emerged from their algorithm, they did succeed in calculating π to approximately two billion places; a world record at the time. Remarkably, π has now been calculated to 22 trillion digits and if you were to print out this file, it would be equivalent to several million books, each containing a thousand pages!

Note 5: Solutions for Activity 3

Q1. Cut out the following two shapes (you may wish to sketch them first!)

(i) A shape with 2 faces and 4 edges

Square, rectangle, rhombus or any irregular 4-sided shape.

(ii) A shape with 2 faces and 1 edge

Circle or oval.

Q2. Can you construct a shape with 1 face and 1 edge using only paper, Sellotape and scissors? Can you prove that your shape only has 1 face and 1 edge using only a pencil?

Allow students to attempt this activity before showing them how to construct a Möbius strip (see video link in **Additional Resources**).

To prove that the Möbius strip has only one face, you can trace a line down its centre until you reach the starting point again (see dotted line in Figure 2). Close inspection will show that this line traversed the entire strip, thereby proving that it only has one face.

Figure 2: Showing that a Möbius strip only has one face and one edge

Similarly, to show that the Möbius strip only has one edge, trace out the edge with a marker and continue around the shape until you reach the starting point again (see Figure 2).

Note 6: Solutions for Activity 4

The Möbius Strip:

Q1. Cut your strip in two, lengthwise along the centre. What do you predict will happen?

You get one long twisted loop with two faces.

Q2. Make a new Möbius strip but this time, try cutting along one-third of its width. What do you predict will happen?

You get two interconnected loops with one larger than the other.

Q3. Make a strip with one full twist and cut it in two, lengthwise along the centre. What do you predict will happen?

You get two full twist loops that are tangled together.

Figure 3: Maths Sparks students and volunteers demonstrating that a Möbius strip only has one face

The Story of Niamh:

Q. What is Niamh's mistake? Please explain your reasoning.

Niamh has miscalculated, but you might enjoy the discussion around this with your students. When we mark a start point and measure the distance walked by Niamh, she walked two kilometres. However, without measuring this specific distance, there is an infinite walkway. Have **fun!**

Note 7: Solutions for Activity 5

Q1. Try making some strips with 2, 3, 4 or 5 half twists (i.e. 180° rotations). How many faces and edges does each one have? Cut these new strips in half and describe what happens!

NUMBER OF HALF TWISTS	NUMBER OF FACES	NUMBER OF EDGES	WHEN CUT IN HALF
1	1	1	1 large loop - 2 faces
2	2	2	2 interconnected loops - 2 faces
3	1	1	1 twisted large loop - 2 faces
4	2	2	2 interconnected loops - 2 faces
5	1	1	1 very twisted large loop - 2 faces

Q2. Do you notice any pattern?

Where there is an odd number of half twists, the resulting shape only has 1 face and 1 edge. When the number of half twists is even, the resulting shape has 2 faces and 2 edges.

Note 8: The Klein bottle

The Klein bottle is a closed, one-sided topological surface that was first proposed by the German mathematician, **Felix Klein**, in 1882 (see Figure 4). It can be formed by gluing two Möbius strips together along their edges, as demonstrated by the video clip included in the **Additional Resources** section. If we were to trace along the surface of the Klein bottle, we would go from what we might perceive as being the 'inside' of the bottle to the 'outside' without actually crossing the boundary between the two. Consequently, we cannot strictly define an inside or outside for the Klein bottle. Nevertheless, the examples of Klein bottles that are commonly depicted in textbooks or on the internet are actually three-dimensional representations of what is, in fact, a four-dimensional surface. A true Klein bottle, for example, does not actually intersect with itself yet this property is impossible to replicate in the three-dimensional models. It is therefore difficult for us to visualise what a true Klein bottle might look like.
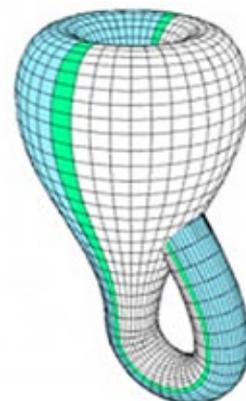

Figure 4: Klein bottle

Figure 5: Doughnut and the Coffee cup.
Image Credit: https://www.shapeways.com

Note 9: The Doughnut and the Coffee Cup

In topology, we look for properties of objects that do not change when they are deformed continuously i.e. without breaking, tearing or gluing. A coffee cup and a doughnut are therefore considered the same from a topological point of view, since both have one hole going all the way through and this property does not change when one is turned into the other (see Figure 5). However, a doughnut is not considered the same as a pretzel since they each have a different number of holes!

Sources and Additional Resources

https://www.youtube.com/watch?v=Cce1XJ3BCxg (Eratosthenes' calculation of the circumference of the Earth)
https://www.geogebra.org/m/zqwhWcfS (Buffon's Needles simulator)
http://www.newyorker.com/magazine/1992/03/02/the-mountains-of-pi (Chudnovsky brothers)
https://brilliant.org/wiki/mobius-strips/ (The Möbius strip)
https://www.youtube.com/watch?v=Z30c5wvoS_s (Constructing a Möbius strip video)
https://plus.maths.org/content/imaging-maths-inside-klein-bottle (Klein bottle)
https://www.youtube.com/watch?v=E8rifKlq5hc (Klein bottle video)

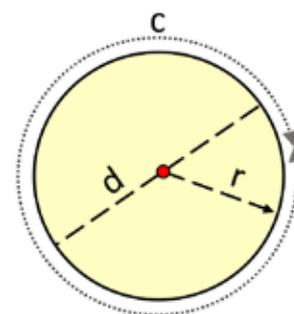# Numbers, Infinity and Shapes: Activity 1

Team Name:

Let's forget about π (pi) and pretend that **Archimedes** never found a value for it. As part of his mathematical team, what might you need to calculate it?

- A metre stick/ruler
- A long piece of string
- A circular object
- A calculator

Task One: Reformulate!

We hypothesise that a relationship exists between the circumference of the circle and the diameter.

c = circumference   **d = diameter**   **r = radius**

Our hypothesis is that for **any circle,** the circumference is equal to some number times the diameter. Let's call this number banana.

Q1 We therefore say that c = banana x d. Rearrange to find banana in terms of c and d

Q2. Repeat, but this time, let d = 2r (i.e. the diameter is twice the radius)

Q3. You should all have a party hat. Let's assume its base is perfectly circular (it should be pretty close!). We want you to calculate the value of banana for this party hat using only a metre stick/ruler, string, your calculator and most importantly, your common sense!

You can use the box below for rough work and then fill your answer into the table.

Once you have finished, you may wish to calculate banana for other circular objects, which you can also fill into the table.

OBJECT	VALUE FOR C	VALUE FOR R	VALUE FOR BANANA

Numbers, Infinity and Shapes: Activity 2 Template

Numbers, Infinity and Shapes: Activity 2

Let's move onto something completely random - literally, random! You should have around 100 assorted wooden sticks per group and a piece of paper with lines on it.

Q1. **In your groups, throw the wooden sticks randomly on the piece of paper and then fill in the summary table below.**

Rough work (optional):

WOODEN STICK LANDS...	TALLY	TOTAL
Crossing a line		
Not crossing a line		
	Total no. of sticks:	

Summary Table:

TOTAL NO. OF STICKS (T)	TOTAL NO. OF STICKS THAT CROSS A LINE (C)	DISTANCE BETWEEN LINES IN MM (D)	LENGTH OF STICKS IN MM (L)

Using the results from his experiment with needles, Buffon estimated the value of using the following formula:

$$\pi \approx \frac{2L}{dx}$$

Where: **L** is the length of the needle (or sticks for us), **d** is the distance between the lines and **x** is the proportion of needles that cross the line (C/T).

Q2. Using this formula, what value did you get for π?

Q3. How accurate is your value for π?

Q4. How could you make your value for π more accurate? Any suggestions?

Q5. Could we ever measure an exact value for π? Explain your reasoning.
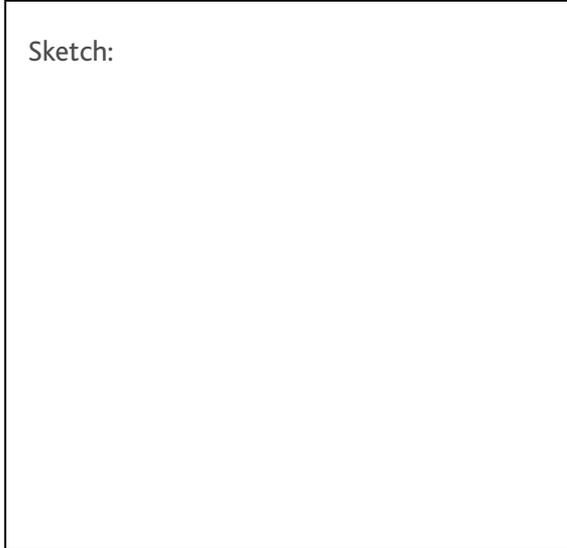
Numbers, Infinity and Shapes: Activity 3

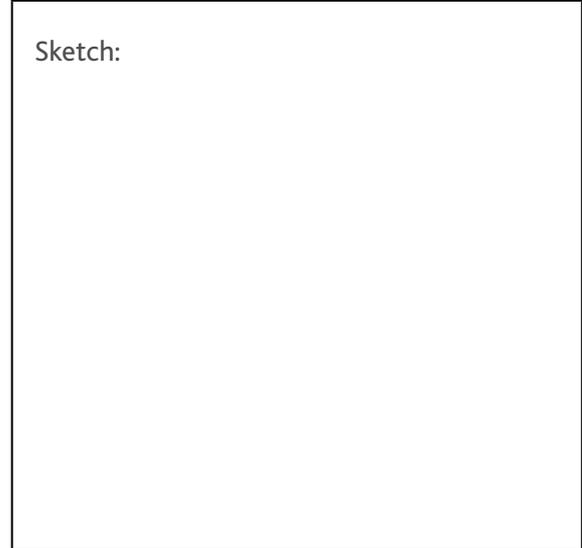**Q1. Cut out the following two shapes (you may wish to sketch them first!)**

(i) A shape with 2 faces and 4 edges

Sketch:

(ii) A shape with 2 faces and 1 edge

Sketch:

Q2. Challenge:

- Can you construct a shape with only 1 face and 1 edge using only paper, Sellotape and scissors?
- Can you prove that your shape only has 1 face and 1 edge using only a pencil?

Q3. Is this shape you constructed 2-D or 3-D?

- If it is 2-D it will have an area, can you measure it?
- If it is 3-D it will have a volume, can you measure it?
- What is the name of this shape? _____

Numbers, Infinity and Shapes: Activity 4

The Möbius Strip

Q1. Cut your strip in two, lengthwise along the centre. What do you predict will happen?

Q2. Make a new Möbius strip but this time, try cutting along one-third of its width. What do you predict will happen?

Q3. Make a strip with one full twist and cut it in two, lengthwise along the centre. What do you predict will happen?

The Story of Niamh

Niamh has been sent to live on the European Space Agency (ESA) space station, which has been constructed in the shape of a Möbius strip. The station has a central path on both the top and bottom surfaces of the loop. One afternoon, Niamh decides to find out just how long the ESA space station actually is and so, starting at a marker on the central path outside the control room, she turns and walks until she arrives back to where she began. Her health app says she has walked 2 kilometres.

Niamh therefore concludes that the space station must have 4 kilometres of central path in total since she knows that there is another path on the opposite surface of the loop.

What is Niamh's mistake? Please tick the relevant box.

☐ Niamh did not make any mistake - she is right!

☐ Niamh has miscalculated - there is actually 1 kilometre of walkway

☐ Niamh has miscalculated - there are actually 2 kilometres of walkway

☐ Niamh has miscalculated - there is actually an infinite length of walkway

Please explain your reasoning below:

_____

_____

_____

_____

_____

_____

Numbers, Infinity and Shapes: Activity 5

NUMBER OF HALF TWISTS	NUMBER OF FACES	NUMBER OF EDGES	WHEN CUT IN HALF
1			
2			
3			
4			
5			

Q1. Try making some strips with 2, 3, 4 or 5 half twists (i.e. 180° rotations). How many faces and edges does each one have? Cut these new strips in half and briefly describe what happens!

_____

_____

Q2. Do you notice any pattern?

_____

_____

_____

_____

_____